M. Andrea A. de Cataldo (2015) "Proper Toric Maps Over Finite Fields,"
International Mathematics Research Notices, Vol. 2015, No. 24, pp. 13106–13121
Advance Access Publication April 1, 2015
doi:10.1093/imrn/rnv094

Proper Toric Maps Over Finite Fields

Mark Andrea A. de Cataldo
Department of Mathematics, Stony Brook University, Stony Brook, NY 11794, USA

Correspondence to be sent to: mark.decataldo@stonybrook.edu

We determine a strong form of the decomposition theorem for proper toric maps over finite fields.

1 Introduction, Notation, Basic Toric Geometry, and Statements

1.1 Introduction

For a general proper map of varieties over a finite field, the decomposition theorem in [1] predicts that the direct image of the intersection cohomology complex becomes semisimple after passage to an algebraic closure of the finite field. In this paper, we prove Theorem 1.1 which establishes that, for proper toric maps of toric varieties, the above semisimplicity already occurs over the finite field, and the simple direct summands are described explicitly. In fact, something stronger is proved, see Remark 2.10, namely that the stalks of the cohomology sheaves of the direct image complex are semisimple Galois-modules, i.e., there are no non-trivial Jordan blocks in the Jordan canonical form of the action of Frobenius on these stalks. Recall this kind of semisimplicity is not known even for the cohomology of smooth projective varieties defined over finite fields.

If the proper toric fibration is surjective with connected fibers, then the direct summands appearing in the decomposition theorem are Tate-twisted intersection cohomology complexes of closures of orbits on the target with constant coefficients; see

Received December 22, 2014; Revised December 22, 2014; Accepted February 25, 2015
Communicated by Corrado de Concini

Theorem 1.1.c. For a general proper toric map f, we first Stein factorize the map $f = h \circ g$, we apply the above to the proper toric fibration g, we push forward each direct summand via the finite toric map h and examine the result; see Theorem 1.1.a and Lemma 2.12.

In order to carry out what above, we first prove Theorem 1.1.b, which here we state in the special case when the domain is smooth: the cohomology of the fibers over closed points is trivial in odd degree, and in even degree is pure with eigenvalues of Frobenius given by a suitable power of the cardinality of the finite field. The triviality in odd degree, for example, implies that, in the context of proper toric maps, a plane cubic with a node may not appear as a fiber of a proper toric map (of course, it may appear as the image of one).

This paper is a companion paper to [2], where the precise form of the decomposition theorem for proper toric varieties over $\mathbb{C}$, as well as an analog of the purity statement above, are proved with different methods. The main purpose of [2] is to then introduce a topological/combinatorial invariant of proper toric fibrations that detects, for example, whether a given orbit contributes a direct summand to the decomposition theorem. This turns out to be related to seemingly subtle combinatorial positivity questions.

The purpose of this paper is also to offer a sample computation in the context of $\overline{\mathbb{Q}}_\ell$-adic cohomology over finite fields in a manner which we hope is accessible to the non-expert. There are two main differences with the situation over $\mathbb{C}$: (1) as mentioned above, in general the conclusion of the decomposition theorem over an algebraic closure of a finite field does not seem to hold in its full-strength over a finite field (see Section 2.1); (2) even if toric maps are defined over $\mathbb{Z}$, the local systems appearing as coefficients in the decomposition theorem depend on the characteristic of the ground field (see Remark 2.13).

1.2 Notation

This paper deals with proper toric maps of toric varieties over a finite field. Our main references are [1, 5].

Toric varieties and toric maps are defined over the ring of integers, hence over any ground field. We view a toric variety $Z = Z(\Delta)$ as the one associated with a fan Δ in a lattice $N \cong \mathbb{Z}^n$ so that $\dim Z = n$. If helpful, we add subscripts: Δ_Z, etc. We view Δ also as a poset: $\tau \leq \sigma$ iff the cone τ is a face of the cone σ iff $V(\tau) \supseteq V(\sigma)$ (reversed inclusion for the closures $V(\tau)$ and $V(\sigma)$ of the orbits $O(\tau)$ and $O(\sigma)$). The support of the fan Δ in $N_{\mathbb{R}}$ is denoted by $|\Delta|$. The n-dimensional torus $T \subseteq Z$ acts on Z with smooth action map $\mathrm{act}: Z \times T \to Z$. Each orbit $O(\sigma)$ carries a distinguished point z_σ which is rational over the prime subfield of the ground field. We denote by Ω the resulting finite partition $Z = \coprod_\sigma O(\sigma)$ into locally closed smooth subvarieties.

A toric map is a toric map $f: X \to Y$ of toric varieties, i.e., the one associated with a linear map $f_N: N_X \to N_Y$ of the lattices with the following property: every cone $\sigma \in \Delta_X$ has image contained inside a cone of Δ_Y. This gives rise to the map of posets $f_\Delta: \Delta_X \to \Delta_Y$, sending $\sigma \mapsto \bar{\sigma} :=$ the smallest cone in Δ_Y containing the image $f_N(\sigma)$. If $\sigma \in \Delta_X$, then one verifies that $f(x_\sigma) = y_{\bar{\sigma}}$.

A toric map is proper iff $f_{N_{\mathbb{R}}}(|\Delta_X|) = |\Delta_Y|$. A proper toric fibration is a proper toric map $f: X \to Y$ such that $f_* \mathcal{O}_X = \mathcal{O}_Y$. If f is proper toric, then f is a proper toric fibration iff f_N is surjective. A proper toric fibration is surjective with connected fibers. A proper toric map which is surjective and with connected fibers is not necessarily a proper toric fibration (e.g., Frobenius).

Unless mentioned otherwise, we work with schemes (separated and of finite type) over a finite field ${}_0\mathbb{F}$, of which we fix an algebraic closure $\mathbb{F}$.

We denote schemes over ${}_0\mathbb{F}$ by using the pre-fix ${}_0-$, which we remove after pulling-back to $\mathbb{F}$: e.g., if ${}_0f: {}_0X \to {}_0Y$ is an ${}_0\mathbb{F}$-map, then we can pull it back to the $\mathbb{F}$-map $f: X \to Y$; similarly, for the complexes below. A standard notation is X_0, etc.; we depart from it for graphical reasons.

We work with the "derived" category of mixed complexes $D^b_m({}_0X, \overline{\mathbb{Q}}_\ell)$, endowed with the middle perversity t-structure [1, p.126, p.101, p.71], whose elements we simply call complexes.

By graded vector space M^*, we mean $\mathbb{Z}$-graded: $M^* = \oplus_{j \in \mathbb{Z}} M^j = M^{\mathrm{even}} \oplus M^{\mathrm{odd}}$. We say that M^* is even if $M^{\mathrm{odd}} = \{0\}$.

Given a finite extension ${}_0\mathbb{F} \subseteq {}_1\mathbb{F} \subseteq \mathbb{F}$, we have the open inclusion of Galois (pro-finite) groups ${}_1G := \mathrm{Gal}(\mathbb{F}/{}_1\mathbb{F}) \subseteq {}_0G := \mathrm{Gal}(\mathbb{F}/{}_0\mathbb{F})$. If ${}_0C$ is a complex on ${}_0X$, then the $\mathbb{Z}$-graded object $H^*(X, C)$ is a continuous (continuity will not be mentioned further) ${}_0G$-module. If $x \in {}_0X(\mathbb{F})$ is a closed point with residue field a finite extension ${}_1\mathbb{F}$, then the graded object $\mathcal{H}^*(C)_x$ is a ${}_1G$-module. The weight-like properties of the cohomology groups and stalks we consider are well-defined independently of the finite field extension one works with; see [1, 5.1.12]. Instead of insisting on ${}_0G$, ${}_1G$, etc., by abuse of notation, we simply talk about G-modules and their weights.

Given a variety ${}_0X$, we have the associated shifted intersection complex $\mathcal{I}_{{}_0X}$; if ${}_0X$ is smooth, then $\mathcal{I}_{{}_0X} = \overline{\mathbb{Q}}_{\ell_{{}_0X}}$. For convenience, we also use the intersection complex $I_{{}_0X} := \mathcal{I}_{{}_0X}[\dim {}_0X]$, which is a perverse sheaf on ${}_0X$. The intersection cohomology groups of a variety ${}_0X$ are $IH^*(X) := H^*(X, \mathcal{I}_X)$.

We have the notions of even, mixed, and pure G-module M^*: e.g., M^* is said to be pure of weight w if M^j is pure of weight $w+j$ for every j.

A graded G-module M^* is said to be Tate if it is even and each $M^{2k} \cong \overline{\mathbb{Q}}_\ell^{\oplus s_k}(-k)$, for some $s_k \in \mathbb{Z}^{\geq 0}$. Note that a Tate module is semisimple (under the action of the Frobenius automorphism).

A complex ${}_0C$ on ${}_0X$ is said to be punctually pure of weight w if the graded G-modules $\mathcal{H}^*(C)_x$ are pure of weight w for every closed point $x \in {}_0X(\mathbb{F})$. Similarly, we have the notion of ${}_0C$ being even, and of ${}_0C$ being Tate. In particular, we have the notion of ${}_0C$ being pure, punctually pure, even and Tate; e.g., see Theorem 1.1.

By a result of Gabber, the intersection complex $\mathcal{I}_{{}_0X}$ of a variety is pure of weight zero. The Tate-shifted $\mathcal{I}_{{}_0X}(-k)$ is pure of weight $2k$, and I_X is pure of weight $\dim X$. By a result of Deligne, if ${}_0f : {}_0X \to {}_0Y$ is a proper map of varieties, then $R\,{}_0f_*\mathcal{I}_{{}_0X}$ is pure of weight zero. However, in general, $\mathcal{I}_{{}_0X}$ and $R\,{}_0f_*\mathcal{I}_{{}_0X}$ are neither punctually pure, even, nor Tate.

A complex on a toric variety is said to be Ω-constructible if its restriction to each orbit has lisse (the notion of lisse sheaf is the $\overline{\mathbb{Q}}_\ell$-adic analogue of a locally constant sheaf) cohomology sheaves. A skyscraper constant sheaf at the origin of the affine line is Ω-constructible, whereas one at the point 1 is not. The intersection complex of a toric variety ${}_0Z$ is Ω-constructible. The direct image complex $R\,{}_0f_*\mathcal{I}_{{}_0X}$ via a proper toric map ${}_0f$ may fail to be Ω-constructible, e.g., the second closed embedding above (the first one is not a toric map, according to our definitions).

Given a toric map ${}_0f : {}_0X \to {}_0Y$ define the cohomology graded sheaf on Y by setting $R^* := \oplus_j R^j f_*\mathcal{I}_X$. Denote the restriction of R^* to an orbit $O(\sigma) \subseteq Y$ by R^*_σ, and its stalk at a closed point $y \in {}_0Y(\mathbb{F})$ by R^*_y, this is a graded G-module. If ${}_0f$ is proper, then proper base change yields $R^*_y = H^*(f^{-1}(y), \mathcal{I}_X)$ (pull-back/restriction symbols are mostly omitted throughout the paper).

1.3 Some basic toric algebraic geometry

In this section, we work over an arbitrary ground field. Let $Z = Z(\Delta)$ be a toric variety and let $f : X \to Y$ be a toric map. What follows is a list of the properties we need; for proofs and/or references, see [2].

1.3.1 *Toric affine open cover, orbit closures, and partial order*

Z is covered by the open affine toric subvarieties $U_\zeta = \coprod_{\rho \leq \zeta} O(\rho)$, where $\zeta \in \Delta$. We have $V(\zeta) := \overline{O(\zeta)} = \coprod_{\rho \geq \zeta} O(\rho)$, and $\tau \leq \sigma$ iff $V(\tau) \supseteq O(\sigma)$.

1.3.2 *Toric varieties of contractible type*

The toric variety Z is said to be of contractible type if it is of the form $(Z, z) = (U_\zeta, z_\zeta)$, where the cone ζ spans $N_{\mathbb{R}}$. In this case, z is the unique torus fixed point.

1.3.3 *The local product structure of Z along orbits*

Given $\zeta \in \Delta$, any splitting $N \cong N_\zeta \oplus N(\zeta)$ of lattices determines a splitting $T \cong T_\zeta \times T(\zeta)$ of tori, and an equivariant isomorphism of toric varieties:

$$U_\zeta \cong U_{\zeta'} \times O(\zeta), \tag{1}$$

where ζ' is the cone ζ, viewed in $N_\zeta \subseteq N$. One virtue of (1) is that $U_{\zeta'}$ is of contractible type and the product assertion is useful in the context of inductive arguments; the same is true for (2). The isomorphism (1) depends—harmlessly, for us—on the choice of the splitting of lattices above. The fan in $N_{\zeta'} := N_\zeta \subseteq N$ given by ζ' and its faces yields a canonical closed embedding $(U_{\zeta'}, z_{\zeta'}) \to (U_\zeta, z_\zeta)$, compatible with the non-canonical (1).

1.3.4 *The local product structure of a proper toric fibration over the U_σ*

Let f be a proper toric fibration. Let $\sigma \in \Delta_Y$. There is a non-canonical equivariant splitting as in (1), and a non-canonical equivariant isomorphism of toric maps, compatible with (1):

$$(f^{-1}(U_\sigma) \to U_\sigma) \cong (f^{-1}(U'_\sigma) \times O(\sigma) \xrightarrow{f_{\sigma'} \times \mathrm{Id}} U_{\sigma'} \times O(\sigma)). \tag{2}$$

The resulting natural restriction-over-$U_{\sigma'}$-map $f_{\sigma'}$ is a toric fibration onto a base of contractible type, and we have a natural identification $f^{-1}(y_\sigma) = f_{\sigma'}^{-1}(y_{\sigma'})$. In particular, we get a $(T_X \to T_Y(\sigma))$-equivariant non-canonical decomposition:

$$f^{-1}(O(\sigma)) \cong f^{-1}(y_\sigma) \times O(\sigma). \tag{3}$$

1.3.5 *Canonical factorization of induced maps between orbits*

Let $\xi \in \Delta_X$ and consider the natural map of tori $\phi : (O(\xi), x_\xi) \to (O(\bar{\xi}), y_{\bar{\xi}})$ induced by f. The image is a closed subtorus $i : (O'(\xi), y_{\bar{\xi}}) \to (O(\bar{\xi}), y_{\bar{\xi}}))$, and there is the following canonical factorization into maps of tori:

$$\phi : O(\xi) \xrightarrow{a} A \xrightarrow{b} B \xrightarrow{c} O'(\xi) \xrightarrow{i} O(\bar{\xi}), \tag{4}$$

where a is a toric fibration (non-canonically, a product projection); b is a geometric quotient map, étale and Galois, by the action of a finite Abelian subgroup of the torus A; c is a universal homeomorphism, i.e., a homeomorphism that stays such after arbitrary base change; i is the natural closed embedding above.

1.3.6 *The Stein factorization of a toric map*

Let $f: X \to Y$ be a proper toric map. There is the canonical toric Stein factorization:

$$f: X \xrightarrow{g} Z \xrightarrow{h} Y, \tag{5}$$

where g is a proper toric fibration ($g_* \mathcal{O}_X = \mathcal{O}_Z$; surjective with connected fibers), and h is toric finite. The normalization of the image $f(X)$ is a toric variety.

1.3.7 *Toric resolutions, toric Chow envelopes, and toric completions*

There is a proper birational toric map $g: W \to X$ such that W is nonsingular; one can choose W to be quasi projective, so that g is then projective. In particular, if f is proper, then there is a projective toric map $g: W \to X$ such that $h := f \circ g$ is projective toric.

There is a toric completion of X, i.e., an open immersion $j: X \to \bar{X}$ such that j is a toric map and $\bar{X}$ is toric complete.

There are toric completions $X \subseteq \bar{X}$ and $Y \subseteq \bar{Y}$ and a proper toric map $\bar{f}: \bar{X} \to \bar{Y}$ extending f.

1.3.8 *Equivariant complexes*

In what follows, we work over a field that is either finite or algebraically closed. Let $\mathrm{act}: Z \times T \to Z$ be the torus action on a toric variety. A complex C on Z is equivariant (the standard definition of equivariance requires the usual cocycle condition; we do not need it here) if there is an isomorphism $\mathrm{act}^* C \cong \pi_Z^* C$. We can extend this notion to the torus-invariant subschemes of Z. The intersection complex $\mathcal{I}_{V(\sigma)}$ of the closure of an orbit is equivariant. Given $\nu \in N$, we have the associated co-character $\lambda_\nu: \mathbb{G}_m \to T$, and the associated notions of λ_ν-equivariance. If $f: X \to Y$ is a proper toric fibration, then Rf_* preserves equivariance.

1.3.9 *The retraction lemma*

This is where the notion of toric variety of contractible type starts playing a role. Let $f: X \to Y$ be a proper toric fibration onto (Y, y) of contractible type, and let C be an equivariant complex on X (the ground field is algebraically closed, or finite). The natural graded map below is an isomorphism:

$$H^*(X, C) = H^*(Y, Rf_*C) \xrightarrow{=} (R^* f_* C)_y, \tag{6}$$

where it is understood that if the ground field is finite, then we have passed to an algebraic closure, and we have an isomorphism of G-modules. Special case: $C = \mathcal{I}_X$; then, by coupling with proper base change:

$$IH^*(X) = (R^* f_* \mathcal{I}_X)_y = H^*(f^{-1}(y), \mathcal{I}_X). \tag{7}$$

Special case of the special case: $f = \mathrm{Id}_Y$; then:

$$IH^*(Y) = \mathcal{H}^*(\mathcal{I}_Y)_y. \tag{8}$$

1.4 The decomposition theorem for proper toric maps over finite fields

Let ${}_0f : {}_0X \to {}_0Y$ be a proper toric map over a finite field ${}_0\mathbb{F}$. Let ${}_0f = {}_0h \circ {}_0g : {}_0X \to {}_0Z \to {}_0Y$ be the Stein factorization. For every $\zeta \in \Delta_{{}_0Z}$, define, recalling (4):

$$\mathrm{Ev}_\zeta := \{b \in \mathbb{Z} \mid b + \dim X - \dim V(\zeta) \text{ even}\}, \qquad \beta_\zeta := \frac{b + \dim X - \dim V(\zeta)}{2},$$

$${}_0O'(\zeta) := {}_0h({}_0O(\zeta)), \qquad {}_0L_\zeta = {}_0h_* \overline{\mathbb{Q}}_{\ell_{{}_0O(\zeta)}} \text{ a lisse sheaf on } {}_0O'(\zeta) \subseteq {}_0O(\bar{\zeta}).$$

Theorem 1.1 (DT for proper toric maps over finite fields)**.**

(a) Let ${}_0f : {}_0X \to {}_0Y$ be a proper toric map. There is a DT isomorphism in $D^b_m({}_0Y, \overline{\mathbb{Q}}_\ell)$:

$$Rf_* I_{{}_0X} \cong \bigoplus_{\zeta \in \Delta_{{}_0Z}} \bigoplus_{b \in \mathrm{Ev}_\zeta} I_{\overline{{}_0O'(\zeta)}}^{s_{\zeta,b}}({}_0L_\zeta)(-\beta_\zeta)[-b], \tag{9}$$

where ${}_0O'(\zeta) := {}_0h({}_0O(\zeta))$; the sheaves ${}_0L_\zeta = {}_0h_* \overline{\mathbb{Q}}_{\ell_{{}_0O(\zeta)}}$ on ${}_0O'(\zeta)$ are lisse, semisimple, and pure of weight zero; the $s_{\zeta,b} \in \mathbb{Z}^{\geq 0}$ are subject to:

(i) $s_{\zeta,b} = s_{\zeta,-b}$, for every $b \in \mathrm{Ev}_\zeta$;

(ii) if ${}_0f$ is projective, then $s_{\zeta,b} \geq \sum_{l \geq 1} s_{\zeta,b+2l}$, for every $b \geq 0$ in Ev_ζ.

(b) In particular, the pure weight zero $R_0 f_* \mathcal{I}_{{}_0X}$ is punctually pure, even and Tate; for every $y \in {}_0Y(\mathbb{F})$, the G-module $(R^* f_* \mathcal{I}_X)_y = H^*(f^{-1}(y), \mathcal{I}_X)$ is pure, even and Tate.

(c) Let ${}_0f$ be a proper toric fibration, for $\sigma \in \Omega_Y$, and let Ev_σ, and β_σ be as above. There is a DT isomorphism in $D^b_m({}_0Y, \overline{\mathbb{Q}}_\ell)$:

$$Rf_* I_{{}_0X} \cong \bigoplus_{\sigma \in \Omega_{{}_0Y}} \bigoplus_{b \in Ev_\sigma} I^{s_{\sigma,b}}_{{}_0V(\sigma)}(-\beta_\sigma)\,[-b], \tag{10}$$

where the $s_{\sigma,b} \in \mathbb{Z}^{\geq 0}$ are subject to the conditions analogous to (i) and (ii) above. □

Remark 1.2. Even though statement (a) implies statements (b) and (c), we prove the three assertions in the following order: assertion (b) is proved by Corollary 2.8, assertion (c) is proved by Theorem 2.11, and assertion (a) is proved by Theorem 2.14, which builds on assertion (c). □

Remark 1.3. We may re-write the DT isomorphisms using the shifted $\mathcal{I}_X$ by setting $2k = 2\beta$, etc. In the case of a proper toric fibration, we get, with $s_{\sigma,b}$ as in (9):

$$R_0 f_* \mathcal{I}_{{}_0X} \cong \bigoplus_{\sigma \in \Delta_{{}_0Y}} \bigoplus_{k \in \mathbb{Z}^{\geq 0}} \mathcal{I}^{s_{\sigma,2k-\dim {}_0X+\dim {}_0V(\sigma)}}_{{}_0V(\sigma)}(-k)[-2k]. \tag{11}$$

While (10) emphasizes duality, (11) emphasizes eveness. □

Remark 1.4 (DT for proper toric maps over algebraically closed fields)**.** Theorem 1.1 over finite fields implies the analogous results over any algebraically closed field (remove ${}_0-$ and $(-\beta)$). In fact, assume that $\mathrm{char}\, K \neq 0$ and form the tower of field extensions ${}_0\mathbb{F} \subseteq \mathbb{F} \subseteq K$ given by the prime subfield of K and by its algebraic closure in K. Theorem 1.1 over a finite field implies immediately the desired conclusions for $K = \mathbb{F}$. One then pulls-back further to K and concludes when $\mathrm{char}\, K \neq 0$.

According to [1, Section 6], especially Section 6.1.10, Lemma 6.2.6 and Theorem 6.2.5, relating the situation over an algebraically closed field of characteristic zero to the one over an algebraic closure of a finite field, the desired conclusion in characteristic zero follows from the one in positive characteristic.

If the ground field is $\mathbb{C}$ and we use the classical topology, then one can reach analogous conclusions by using Saito's theory of mixed Hodge modules. □

2 Decomposition Theorem for Proper Toric Maps Over Finite Fields

2.1 General DT package over finite fields

The following is surely well known and follows easily from some of the results in [1]. We could not find an adequate explicit reference.

Proposition 2.1 (DT and RHL over ${}_0\mathbb{F}$). Let ${}_0f: {}_0X \to {}_0Y$ be a proper map of separated ${}_0\mathbb{F}$-schemes of finite type, let ${}_0P$ be a pure perverse sheaf of weight w on ${}_0X$. Then the direct image complex $R_0f_{*0}P$ is pure of weight w and splits non-canonically into the direct sum $\bigoplus_b {}^pH^b(R_0f_{*0}P)[-b]$ of its shifted perverse direct image complexes; the perverse sheaves ${}^pH^b(R_0f_{*0}P)$ are pure of weight $w+b$ and admit the canonical decomposition by supports as the direct sum $\bigoplus_{{}_0\mathfrak{Y}} IC_{{}_0\mathfrak{Y}}({}_0L_{b,{}_0\mathfrak{Y}})$ of finitely many intersection complexes of ${}_0\mathbb{F}$-integral subvarieties ${}_0\mathfrak{Y} \subseteq {}_0Y$, with coefficients pure lisse sheaves ${}_0L_{b,{}_0\mathfrak{Y}}$ of weight $w+b-d_{{}_0\mathfrak{Y}}$ on suitable Zariski dense open subsetes ${}_0\mathfrak{Y}^o \subseteq {}_0\mathfrak{Y}$.

Assume, in addition, that ${}_0f$ is projective and let ${}_0\eta$ be the first Chern class of an ${}_0f$-ample line bundle on ${}_0X$. Then the relative hard Lefschetz theorem (RHL) holds: for every $i \geq 0$, the cup product map ${}_0\eta^i : {}^pH^{-i}(R_0f_{*0}P) \longrightarrow {}^pH^i(R_0f_{*0}P)(i)$ on the perverse cohomology sheaves of the direct image is an isomorphism. □

Proof. By virtue of Deligne's fundamental result [1, 5.1.14], the direct image $R_0f_{*0}P$ is pure of weight w. The perverse sheaves ${}^pH^b(R_0f_{*0}P)$ are pure of weight $w+b$ by [1, Theorem 5.4.1]. They split as indicated by virtue of [1, Corollary 5.3.11], coupled with a straightforward Noetherian induction.

In the projective case, the RHL is [1, Theorem 5.4.10], and the splitting into the direct sum of shifted perverse cohomology sheaves is a formal consequence of RHL via the Deligne–Lefschetz criterion.

The DT for a proper map can be derived formally as follows.

By using [1, Corollary 5.3.11], we first observe that we may assume that ${}_0P$ is the intermediate extension back to ${}_0X$ of its own restriction to any Zariski-dense open subvariety.

We choose a Chow envelope ${}_0h: {}_0W \xrightarrow{{}_0g} {}_0X \xrightarrow{{}_0f} {}_0Y$ of the map ${}_0f$, so that ${}_0g$ and ${}_0h$ are projective and there is a Zariski dense open subvariety ${}_0U \subseteq {}_0X$ over which ${}_0g$ is an isomorphism.

Let ${}_0P'$ be the corresponding intermediate extension to ${}_0W$. Since ${}_0g$ is projective, we can apply the Deligne–Lefschetz splitting and deduce that ${}_0P$ is a direct summand of $R_0g_{*0}P'$.

The conclusion follows by applying what we have already proved for the projective ${}_0g$ and to ${}_0h$, and then by using the relation $R_0h_* = R_0f_* \circ R_0g_*$ to compare terms. ■

Remark 2.2. The lisse sheaves ${}_0L$ on ${}_0\mathfrak{Y}^o$ become semisimple, after pull-back to each integral component of $\mathfrak{Y}^o$. As we shall see, in the case of proper toric maps with ${}_0P := I_{{}_0X}$, the subvarieties ${}_0\mathfrak{Y}$, being torus-orbits, are geometrically integral and, in addition, the lisse coefficients ${}_0L$ turn out to be are already semisimple. See Theorems 2.11 and 2.14. The indecomposable pure perverse sheaves on a ${}_0Y$ are described in [1, Proposition 5.3.9]. Due to my ignorance, even in the case when ${}_0P := I_{{}_0X}$, I do not know if the pure lisse ${}_0L$ may admit indecomposable direct summands that present Jordan block-type factors of type E_n with $n \geq 2$: that would be the only obstacle to the semisimplicity of the pure lisse coefficients ${}_0L$ on ${}_0Y$. □

2.2 First toric consequences of Proposition 2.1

Lemma 2.3. Let ${}_0Z \stackrel{{}_0u}{\to} {}_0U \stackrel{{}_0j}{\to} {}_0V$ be maps of schemes (separated and of finite type) over a finite field and let ${}_0C$ be a pure complex of weight w on ${}_0V$. Assume that ${}_0Z$ is complete, ${}_0j$ is an open immersion, and $u^* : H^*(U, C) \to H^*(Z, C)$ is an isomorphism, ${}_0V$ is smooth. If the—automatically pure—submodule of lowest weight w of $H^*(V, C)$ is even and Tate, then so is $H^*(U, C)$. □

Proof. This is standard; we freely use basic weight theory [1, 5.1.14]. It is enough to show that j^* is surjective. We have that $H^*(U, C) \cong H^*(Z, C)$ has weights $\leq w$, because ${}_0Z$ is complete, so that the direct image coincides with the extraordinary direct image, under which the property of having weights $\leq w$ is stable. $H^*(U, C)$ has weights $\geq w$ because the property of having weights $\geq w$ is stable under direct image. By combining the two weight inequalities above, we see that $H^*(U, C)$ is pure of weight w. Let ${}_0i$ the closed embedding complementary to ${}_0j$. In view of the fact that the property of having weight $\geq w$ is stable under extraordinary inverse image, the long exact sequence of relative cohomology $H^*(V, C) \to H^*(U, C) \to H^{*+1}(V, i_!i^!C)$ shows that j^* is surjective. ■

Lemma 2.4. Let ${}_0X$ be a toric variety over a finite field. The intersection complex $\mathcal{I}_{{}_0X}$ is pure of weight zero and Ω-constructible. □

Proof. Both are well known. We offer a proof of purity for toric varieties based on the DT. Of course, purity of the intersection complex for any variety over a finite field is a result of Gabber. We also prove Ω-constructibility as we need some of the details of the proof in the proof of Lemma 2.5.

Let ${}_0g : {}_0W \to {}_0X$ be a proper toric resolution of the singularities of ${}_0X$. Proposition 2.1 implies that $\mathcal{I}_{{}_0X}$, being a direct summand of the pure weight zero $R_0 g_* \overline{\mathbb{Q}}_{\ell_0 W}$, is pure of weight zero.

The proof of Ω-constructibility is by induction on $n := \dim {}_0X$. If $n = 0$, then we are done. Assume the desired conclusion holds for every toric variety of dimension at most $n-1$.

Since the open sets of the form ${}_0U_\sigma$ are union of orbits and cover ${}_0X$ as σ ranges in Δ_X, we may assume that ${}_0X = {}_0U_\sigma$. In view of the local product structure ${}_0U_\sigma \cong {}_0U_{\sigma'} \times {}_0O(\sigma)$, we may also assume that $({}_0X, {}_0x)$ is of contractible type.

Since ${}_0x$ is now an orbit, we may replace $({}_0X, {}_0x)$ with ${}_0X \setminus {}_0x$. Now, ${}_0X$ is covered by the affine open sets of the form ${}_0U_\tau \cong {}_0U_{\tau'} \times {}_0O(\tau)$, with $\tau < \sigma$, so that $\dim {}_0U_{\tau'} \leq n-1$, and we are done. ■

Lemma 2.5. Let ${}_0f : {}_0X \to {}_0Y$ be a proper toric fibration over a finite field. The direct image complex $R_0 f_* \mathcal{I}_{{}_0X}$ is Ω-constructible. □

Proof. As in the proof of Lemma 2.4, we may assume that ${}_0Y = {}_0U_\sigma$.

According to (2), the intersection complex of ${}_0f^{-1}({}_0U_\sigma)$ is a pull-back from the factor ${}_0f^{-1}({}_0U_{\sigma'})$. It follows that we may assume that $({}_0Y, {}_0y)$ is of contractible type.

We conclude the proof by arguing by induction on the dimension of the base of contractible type as in the proof of Lemma 2.4. ■

Lemma 2.6. Let ${}_0f : {}_0X \to {}_0Y$ be a proper toric fibration onto a base of contractible type $({}_0Y, {}_0y)$, all over a finite field. Then we have natural isomorphisms of graded G-modules:

$$IH^*(X) = (R^* f_* \mathcal{I}_X)_y = H^*(f^{-1}(y), \mathcal{I}_X). \tag{12}$$

In particular, we have the natural isomorphism of graded G-modules:

$$IH^*(Y) = \mathcal{H}^*(\mathcal{I}_Y)_y. \tag{13}$$

The graded G-modules (12) and (13) above are pure, even and Tate, hence semisimple. □

Proof. The first two assertions follow from the retraction lemmas (7) and (8) and proper base change.

We turn to the proof that the G-modules (12) are pure even and Tate.

By taking a projective toric resolution of ${}_0X$ as in the proof of Lemma 2.4, we may assume that ${}_0X$ is smooth and quasi projective. In particular, we are now dealing with cohomology, instead of intersection cohomology.

Choose a toric open embedding $j: {}_0X \to \overline{{}_0X}$ with $\overline{{}_0X}$ toric smooth and projective.

It is well known that the graded G-module $H^*(\overline{{}_0X})$ is pure of weight zero, even and Tate: it is generated by algebraic cycle classes [5].

We conclude by applying Lemma 2.3 to ${}_0f^{-1}({}_0y) \to {}_0X \to \overline{{}_0X}$. ∎

Remark 2.7 (Description of $H^*(f^{-1}(y), \mathcal{I}_X)$, f proper toric)**.** Let $f: X \to Y$ be a proper toric map over an algebraically closed field and let $y \in Y$ be a closed point. Let $f = h \circ g$: $X \to Z \to Y$ be the toric Stein factorization. The fiber $f^{-1}(y)$ is a disjoint union of the finitely many fibers $g^{-1}(z_l)$, with $\{z_l\} = h^{-1}(y)$. Clearly, $H^*(f^{-1}(y), \mathcal{I}_X) = \oplus_l H^*(g^{-1}(z_l), \mathcal{I}_X)$. Fix z_l and take the $U(\zeta)$ with $z \in O(\zeta)$. By using the local product structure (2) over U_ζ, we may assume that $z_l = z_\zeta$ and that (Z, z_ζ) is of contractible type. By combining the proofs of lemmata 2.6 and 2.4, we obtain the following description:

$H^*(f^{-1}(y), \mathcal{I}_X)$ is a finite direct sum of subquotients of the graded vector spaces $H^*(\bar{W}_l, \overline{\mathbb{Q}}_\ell)$ given by the cohomology of nonsingular projective toric varieties $\bar{W}_l$.

The description remains valid in the context of G-modules if the ground field is the algebraic closure of a finite field. Similarly, over $\mathbb{C}$, in the context of the classical Euclidean topology with the rational mixed Hodge structures of the theory of mixed Hodge modules. □

Corollary 2.8. Let ${}_0f: {}_0X \to {}_0Y$ be a proper toric map over a finite field. Then the pure weight zero complexes $\mathcal{I}_{{}_0X}$ and $R_0 f_* \mathcal{I}_{{}_0X}$ are punctually pure, even, and Tate. □

Proof. The second statement implies the first one by taking ${}_0f = \mathrm{Id}_{{}_0X}$.

Since the desired conclusions are stable under direct images via finite toric maps, in view of the toric Stein factorization of the map ${}_0f$, we may assume that ${}_0f$ is a proper toric fibration.

Since the desired conclusions are statements about the stalks and the direct image complex is Ω-constructible, it is enough to verify that for every distinguished point ${}_0y_\sigma$ of any orbit ${}_0O(\sigma) \subseteq {}_0Y$, the graded G-module $(R^* f_* \mathcal{I}_X)_{y_\sigma}$ is pure, even and Tate. This follows immediately from Lemma 2.6. ∎

Remark 2.9. The punctual purity, etc., of the intersection complex of a toric variety is proved in [4]. The statement for the direct image seems new. □

Remark 2.10. Lemma 2.6 tells us that the stalks of the direct image $R_0 f_* I_{0X}$ are semisimple G-modules. This additional semisimplicity aspect, which should not be confused with the semisimplicity of the direct image $R_0 f_* I_{0X}$, is explored and amplified in the context of the geometry of Schubert varieties in the paper [3]. □

2.3 Proof of Theorem 1.1.c on proper toric fibrations

The following theorem proves Theorem 1.1.c.

Theorem 2.11 (Proper toric fibrations: DT semisimplicity and RHL over ${}_0\mathbb{F}$)**.**

Let ${}_0f : {}_0X \to {}_0Y$ be a proper toric fibration. There are isomorphisms in $D^b_m({}_0Y, \overline{\mathbb{Q}}_\ell)$:

$$R_0 f_* I_{0X} \cong \bigoplus_{\sigma \in \Delta_{0Y}} \bigoplus_{b \in \mathbb{Z}} {}_0J_{\sigma,b}[-b], \tag{14}$$

$${}_0J_{\sigma,b} \cong \begin{cases} 0 & b + \dim X - \dim V(\sigma) \text{ odd}, \\ {}_0I_\sigma^{s_{\sigma,b}} \left(-\dfrac{b + \dim X - \dim V(\sigma)}{2} \right) & b + \dim X - \dim V(\sigma) \text{ even}. \end{cases} \tag{15}$$

The integers $s_{\sigma,b}$ are subject to the following relations:

(i) $s_{\sigma,b} = s_{\sigma,-b}$ (Poincaré–Verdier duality); if $b + \dim X - \dim V(\sigma)$ is odd, then $s_{\sigma,b} = 0$.

(ii) If ${}_0f$ is projective, then for every $b \geq 0$, $s_{\sigma,b} \geq \sum_{l \geq 1} s_{\sigma,b+2l}$ (RHL). □

Proof. Proposition 2.1, coupled with the Ω-constructibility of the direct image complex in Lemma 2.4, implies the existence of an isomorphism (14), where each ${}_0J_{b,\sigma}$ is of the form ${}_0I_{V(\sigma)}({}_0L_{b,\sigma})$, with ${}_0L_{b,\sigma}$ lisse, pure of weight $b + \dim X - \dim V(\sigma)$ on ${}_0O(\sigma)$.

By Ω-constructibility, for each j, σ, the mixed sheaf ${}_0R^j_\sigma$ (see the end of Section 1.2 on notation) is lisse on the orbit ${}_0O(\sigma)$.

The local product structure of ${}_0f$ (3) implies that ${}_0R^j_\sigma$ is given by the continuous representation:

$$\rho^j_\sigma : \pi_1({}_0O(\sigma), y_\sigma) \longrightarrow \pi_1({}_0y_\sigma, y_\sigma) \longrightarrow \mathrm{GL}_{\overline{\mathbb{Q}}_\ell}(R^j_{Y_\sigma}), \tag{16}$$

where the first homomorphism stems from the natural constant ${}_0\mathbb{F}$-map ${}_0O(\sigma) \to {}_0Y_\sigma$, and the second one from the G-module structure on $R^j_{Y_\sigma}$: this means that the lisse sheaf ${}_0R^j_\sigma$ is the pull-back of a lisse sheaf on ${}_0Y_\sigma$, namely, the G-module $R^j_{Y_\sigma}$.

By combining with Corollary 2.8, we see that ${}_0R^j_\sigma$ is zero for j odd and that, for j even, we have ${}_0R^j_\sigma \cong \overline{\mathbb{Q}}^r_{\ell_{0}O(\sigma)}(-j/2)$, for some $r \geq 0$.

The already-proved existence of an isomorphism (14) as above implies that ${}_0L_{b,\sigma}$ is a direct summand of ${}_0R^j_\sigma$ for $j = b + \dim X - \dim V(\sigma)$, so that we have ${}_0L_{b,\sigma} \cong \overline{\mathbb{Q}}^s_{\ell_{0}O_\sigma}(-j/2)$, for some integer s, equal to zero if j is odd. This proves (15), as well as the second part of assertion (i).

Finally, given (14) and (15), assertion (ii) and the first part of (i) are immediate consequences of the relative hard Lefschetz theorem, and of Verdier duality, respectively. ■

2.4 Proof of Theorem 1.1.a on proper toric maps

Let ${}_0f : {}_0X \to {}_0Y$ be a proper toric map over a finite field ${}_0\mathbb{F}$ and let ${}_0f = {}_0h \circ {}_0g$ its Stein factorization. Since ${}_0g$ is a proper toric fibration, i.e., the subject of Theorem 2.11, we turn our attention to the finite map ${}_0h : {}_0Z \to {}_0Y$, with image ${}_0f({}_0X)$, a closed subvariety of ${}_0Y$.

Given $\zeta \in \Delta_{{}_0Z}$, we have the map of tori ${}_0h(\zeta) : {}_0O(\zeta) \to O(\bar{\zeta})$, with image ${}_0O'(\zeta) \subseteq O(\bar{\zeta})$. We denote the evident resulting finite map on the closures by $\overline{{}_0h(\zeta)} : \overline{{}_0O(\zeta)} \to \overline{{}_0O'(\zeta)}$.

Lemma 2.12. The $\overline{\mathbb{Q}}_\ell$-adic sheaf ${}_0L_\zeta := {}_0h(\zeta)_*\overline{\mathbb{Q}}_{\ell_{0}O(\zeta)}$ on ${}_0O'(\zeta)$ is lisse, semisimple, pure, punctually pure of weight zero, with eigenvalues of geometric Frobenius equal to 1; the geometric monodromy has eigenvalues roots of unity.

The direct image $R_0h_{*0}\mathcal{I}_{V(\zeta)} = \mathcal{I}_{{}_0O'(\zeta)}({}_0L_\zeta)$ is pure of weight zero, punctually pure, even, and Tate. □

Proof. The map of tori ${}_0h(\zeta)$ admits the canonical factorization (4). Recalling that the map being factored is finite: the map ${}_0a$ must be the identity. The map ${}_0b$ is a quotient by a finite abelian group, $\Gamma := \operatorname{Ker} {}_0b$, whose order is not divisible by $\operatorname{char} {}_0\mathbb{F}$; in particular it is étale, and Galois. The map ${}_0c$ is a universal homeomorphism. The map ${}_0i$ is the evident closed embedding.

Since ${}_0a$, ${}_0c$ and ${}_0i$ are universal homeomorphisms onto their image, they do not effect the direct image calculations and can be ignored.

It follows that ${}_0L_\zeta$ is naturally identified the ordinary direct image sheaf ${}_0b_*\overline{\mathbb{Q}}_\ell$, with ${}_0b$ the ètale quotient by the action of the finite abelian group Γ. All the listed properties of ${}_0L_\zeta$ follows easily from this description.

The equality statement about the direct image complex follows by observing that we have the following natural identifications:

$$R_0h_*\mathcal{I}_{{}_0V(\zeta)} = {}_0h_*\mathcal{I}_{{}_0V(\zeta)} = \overline{{}_0h(\zeta)}_*\mathcal{I}_{{}_0V(\zeta)} = \mathcal{I}_{\overline{{}_0O'(\zeta)}}({}_0L_\zeta) \tag{17}$$

the first one is because h is finite; the second one is because the third term is merely a rewriting of the second; the third identification follows from the fact that the direct image under the finite map $\overline{{}_0h(\zeta)}$ is t-exact for the middle perversity t-structure and hence preserves intersection complexes with twisted coefficients; clearly, the coefficients of the direct image can be read on a Zariski-dense open subset, so that they are given by ${}_0L_\zeta$.

Finally, the last statement follows by the just-established equality and from Corollary 2.8, applied to ${}_0X := {}_0V(\zeta)$: for the properties in question are stable under finite direct image. ■

Remark 2.13. The rank of the local system ${}_0L_\zeta$ is the cardinality of the abelian group Γ, which depends on the characteristic of the finite field ${}_0\mathbb{F}$. □

We can now show that Theorem 2.11 for proper toric fibrations over ${}_0\mathbb{F}$ has the following natural counterpart for proper toric maps over ${}_0\mathbb{F}$, which, in turn, establishes Theorem 1.1.a.

Theorem 2.14 (Proper toric maps: DT semisimplicity and RHL over ${}_0\mathbb{F}$)**.** Let ${}_0f : {}_0X \to {}_0Y$ be a proper toric map over the finite field ${}_0\mathbb{F}$. There are isomorphisms in $D^b_m({}_0Y, \overline{\mathbb{Q}}_\ell)$:

$$R_0 f_* I_{{}_0X} \cong \bigoplus_{\zeta \in \Delta_{{}_0Z}} \bigoplus_{b \in \mathbb{Z}} \overline{{}_0J_{\zeta,b}}[-b], \tag{18}$$

$$\overline{{}_0J_{\zeta,b}} \cong \begin{cases} 0, & b + \dim X - \dim V(\zeta) \text{ odd}, \\ {}_0IC^{s_{\zeta,b}}_{\overline{{}_0O'(\zeta)}}({}_0L_\zeta)\left(-\dfrac{b + \dim X - \dim V(\zeta)}{2}\right), & b + \dim X - \dim V(\zeta) \text{ even}. \end{cases} \tag{19}$$

The complex $R_0 f_* I_{{}_0X}$ is pure of weight zero, punctually pure, even, and Tate.

The integers $s_{\zeta,b}$ are subject to the following relations:

(i) $s_{\zeta,b} = s_{\zeta,-b}$ (Poincaré-Verdier duality); if $b + \dim Z - \dim V(\zeta)$ is odd, then $s_{\zeta,b} = 0$.

(ii) If ${}_0f$ is projective, then for every $b \geq 0$, $s_{\zeta,b} \geq \sum_{l \geq 1} s_{\zeta,b+2l,\zeta}$ (RHL). □

Proof. We have the Stein factorization ${}_0f = {}_0h \circ {}_0g$. We first apply Theorem 2.11, (15) to the proper fibration ${}_0g$. We form the $R_0 g_*$ of each resulting direct summand. We apply Lemma 2.12, which remains valid also after arbitrary Tate twists, to each resulting term. ■

Acknowledgements

The author thanks the Max Planck Institute for Mathematics in Bonn, and the Institute for Advanced Study in Princeton for providing ideal working conditions during the preparation of this paper, and he also thanks T. Haines, L. Li, L. Migliorini, and M. Mustaţă for very useful exchanges.

Funding

Partially supported by NSF grant DMS-1301761 and by a grant from the Simons Foundation (#296737 to Mark Andrea de Cataldo).

References

[1] Beilinson, A. A., J. Bernstein, and P. Deligne. “Faisceaux Pervers.” *Analysis and Topology on Singular Spaces, I (Luminy, 1981)*, 5–171, Astérisque 100. Paris: Société Mathématique de France, 1982.

[2] de Cataldo, M., L. Migliorini, and M. Mustaţă. “The combinatorics and topology of proper toric maps.” (2014): preprint.

[3] de Cataldo, M., T. Haines, and L. Li. “Frobenius semisimplicity in the decomposition theorem for convolution morphisms (working title)” (in preparation).

[4] Denef, J. and F. Loeser. “Weights of exponential sums, intersection cohomology, and Newton polyhedra.” *Inventiones Mathematicae* 106, no. 2 (1991): 275–94.

[5] Fulton, W. *Introduction to Toric Varieties*. Annals of Mathematics Studies 131. Princeton, NJ: The William H. Rover Lectures in Geometry, Princeton University Press, 1993.